$\mathcal{A}$ - equivalence and the Equivalence of Sections of Images and Discriminants

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In [Mo2] and [MM] Mond and Marar obtain a formula relating the $\mathcal{A}_e$- codimension of map germs f_0: $\mathbb{C}^2,0 \longrightarrow \mathbb{C}^3,0$ to the Euler characteristic of the image of a stable perturbation f_t of f_0. This has been proven to hold quite generally for such map germs by de Jong and Pellikaan (unpublished) and by de Jong and van Straten [JS]. One curious aspect of this formula is the presence of the $\mathcal{A}_e$- codimension, which seems to have little relation with the image of f_t. This codimension is related by de Jong and van Straten to the dimension of the space of deformations of $X = \text{Image}(f_0)$ for which the singular set of X deforms flatly. Their arguments depend strongly upon X being a surface singularity in $\mathbb{C}^3$.

In this paper, we derive another relation between $\mathcal{A}$- equivalence and properties of image(f_0). This relation is valid for all dimensions and directly relates the $\mathcal{A}_e$- codimension of f_0 with a codimension of a germ defining Image(f_0) as a section of the image of a stable germ.

diagram 1

$$\begin{array}{ccc} \mathbb{C}^{n'},0 & \xrightarrow{\quad F \quad} & \mathbb{C}^{p'},0 \\ \uparrow & & \uparrow \; g_0 \\ \mathbb{C}^{n},0 & \xrightarrow{\quad f_0 \quad} & \mathbb{C}^{p},0 \end{array}$$

We recall that by Mather [M-IV], if f_0: $\mathbb{C}^n,0 \longrightarrow \mathbb{C}^p,0$ is a holomorphic germ of finite singularity type (i.e. finite contact codimension) then there is a stable germ F: $\mathbb{C}^{n'},0 \longrightarrow \mathbb{C}^{p'},0$ and a germ of an immersion g_0: $\mathbb{C}^p,0 \longrightarrow \mathbb{C}^{p'},0$ with g_0 transverse to F such that f_0 is obtained as a pull-back in diagram 1 (F is the stable unfolding of f_0 [M-IV]).

The germ g_0 has been used to determine $\mathcal{A}$-determinacy properties of f_0 by Martinet [Ma2] and topological determinacy properties by du Plessis [DP]. However, there was lacking a precise relation between equivalence for the germ g_0 and the $\mathcal{A}$- equivalence of f_0. In this paper we derive such a relation.

Let V = D(F) denote the discriminant of F (which is also Image(F) when $n' < p'$). Given a variety-germ $V,0 \subset \mathbb{C}^{p'},0$ there is a notion of "contact equivalence preserving V" on

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germs h: $\mathbb{C}^m,0 \longrightarrow \mathbb{C}^{p'},0$, defined by the action of a group $\mathcal{K}_V$ [D2].

The main results here concern the relation between $\mathcal{K}_V$ - equivalence for g_0 and $\mathcal{A}$- equivalence for f_0. They are:

1) g_0 has finite $\mathcal{K}_V$ - codimension if and only if f_0 has finite $\mathcal{A}$- codimension;

2) if we denote the extended tangent spaces to the $\mathcal{A}$- orbit of f_0 and the $\mathcal{K}_V$ - orbit of g_0 by $T\mathcal{A}_e \cdot f_0$ and $T\mathcal{K}_{V,e} \cdot g_0$, with associated normal spaces

$$N\mathcal{A}_e \cdot f_0 = \theta(f_0)/T\mathcal{A}_e \cdot f_0 \quad \text{and} \quad N\mathcal{K}_{V,e} \cdot g_0 = \theta(g_0)/T\mathcal{K}_{V,e} \cdot g_0$$

then these normal spaces are isomorphic as $\mathcal{O}_{\mathbb{C}^p,0}$-modules (theorem 2);

3) taking dimensions in (2) we obtain (theorem 1)

$$\mathcal{A}_e\text{-codimension}(f_0) = \mathcal{K}_{V,e}\text{-codimension}(g_0).$$

4) if we replace the germ f_0 and the stable germ F by multi-germs f_0: $\mathbb{C}^n,S \longrightarrow \mathbb{C}^p,0$ and F: $\mathbb{C}^{n'},S \longrightarrow \mathbb{C}^{p'},0$ with f_0 finitely determined and F stable then 1) - 3) remain valid (see theorem 3; however, to keep notation simple we give the proofs for the case where $|S|=1$ and observe that they work for all finite S).

The third result allows us to place the Mond-Marar formula into a common context with other formulas which relate the algebraic codimension of (nonlinear) sections of varieties to Euler characteristics of their perturbations.

As corollaries of these results and their proofs we obtain: i) sufficient conditions for unfoldings of f_0 to be $\mathcal{A}$- trivial in terms of the corresponding unfoldings of g_0 being $\mathcal{K}_V$-trivial (but it is unknown whether the converse holds); ii) a proof that unfoldings of f_0 are $\mathcal{A}$-versal if and only if the corresponding unfoldings of g_0 are $\mathcal{K}_V$ - versal and iii) a characterization of the versality discriminant as the set of points where g_0 fails to be transverse to V and an explicit method for computing the versality discriminant for unfoldings of hypersurfaces.

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§1 $\mathcal{A}$ and $\mathcal{K}_V$-equivalence

Here we recall several basic properties of $\mathcal{A}$ and $\mathcal{K}_V$-equivalence; while those of $\mathcal{A}$-equivalence are generally well-known, those of $\mathcal{K}_V$-equivalence are less so. The key properties of these groups which we are interested in are: their tangent spaces and infinitesimal conditions for versality, infinitesimal conditions for triviality of unfoldings, and geometric characterizationsoffinitedeterminacy.

All germs which we consider will be holomorphic. The two principal notions of equivalence for map germs are $\mathcal{A}$ and $\mathcal{K}$-equivalence. We denote the space of holomorphic germs $f_0 : \mathbb{C}^s,0 \to \mathbb{C}^t,0$ by $\mathcal{C}_{s,t}$ and use local coordinates x for $\mathbb{C}^s$ and y for $\mathbb{C}^t$. With $\mathcal{D}_n$ denoting the group of germs of diffeomorphisms $\varphi : \mathbb{C}^n,0 \to \mathbb{C}^n,0$, the group $\mathcal{A} = \mathcal{D}_s \times \mathcal{D}_t$ acts on $\mathcal{C}_{s,t}$ by $(\varphi,\psi)\cdot f_0 = \psi \circ f_0 \circ \varphi^{-1}$. The group $\mathcal{K}$ (contact equivalence) consists of $H \in \mathcal{D}_{s+t}$ such that there is an $h \in \mathcal{D}_s$ so that $H \circ i = i \circ h$ and $\pi \circ H = h \circ \pi$, where $i(x) = (x,0)$ is the inclusion $i : \mathbb{C}^s \hookrightarrow \mathbb{C}^{s+t}$ and $\pi(x,y) = x$ is the projection $\pi : \mathbb{C}^{s+t} \to \mathbb{C}^s$. Then, $\mathcal{K}$ acts on $\mathcal{C}_{s,t}$ by

$$(h(x), H\cdot f(x)) = H(x, f(x)),$$

i.e. $\mathrm{graph}(H\cdot f) = H(\mathrm{graph}(f))$. Germs are $\mathcal{A}$ or $\mathcal{K}$-equivalent if they lie in common orbits of the group actions.

An unfolding of f_0 is a germ $f : \mathbb{C}^{s+q},0 \to \mathbb{C}^{t+q},0$ of the form $f(x,u) = (\bar{f}(x,u),u)$ with $\bar{f}(x,0) = f_0(x)$ (here u denotes local coordinates for $\mathbb{C}^q,0$). Both $\mathcal{A}$ and $\mathcal{K}$ extend to actions on unfoldings: if $\varphi \in \mathcal{D}_{s+q}$ and $\psi \in \mathcal{D}_{t+q}$ are unfoldings then $(\varphi,\psi)\cdot f = \psi \circ f \circ \varphi^{-1}$ while if $H \in \mathcal{D}_{s+t+q}$ is an unfolding with an unfolding $h \in \mathcal{D}_{s+q}$ so that $H \circ i' = i' \circ h$ and $\pi' \circ H = h \circ \pi'$ (and i' and π' are inclusions and projections for $\mathbb{C}^{s+q}$ and $\mathbb{C}^{s+t+q}$). Then

$$(\bar{h}(x,u), H\cdot f(x,u)) \;=\; H(x,\bar{f}(x,u), u).$$

If $(V,0) \subset \mathbb{C}^t,0$ is a germ of a variety then we can define a subgroup of $\mathcal{K}$

$$\mathcal{K}_V \;=\; \{H \in \mathcal{K} : H(\mathbb{C}^s \times V) \subseteq \mathbb{C}^s \times V\}$$

and similarly for unfoldings. This yields $\mathcal{K}_V$-equivalence. Just as $\mathcal{K}$-equivalence captures the equivalence of the germs of varieties $f_0^{-1}(0)$, so too $\mathcal{K}_V$-equivalence captures the equivalence of the germs of varieties $f_0^{-1}(V)$.

For $\mathcal{G} = \mathcal{A}, \mathcal{K}$ or $\mathcal{K}_V$, we say that an unfolding f of f_0 is a *$\mathcal{G}$-trivial unfolding* if it is $\mathcal{G}$-equivalent to the trivial unfolding $f_0 \times id_{\mathbb{C}^q}$. It is *$\mathcal{G}$-trivial as a family* if the $\mathcal{G}$-equivalence preserves the origin for all parameter values. If $f_1(x,u,v) = (\bar{f}_1(x,u,v), u,v)$ is an unfolding of f_0 so that $\bar{f}_1(x,u,0) = \bar{f}_1(x,u)$, then f_1 will be said to extend f. An *extension* f_1 of f *is $\mathcal{G}$-trivial* if it is $\mathcal{G}$-equivalent to $f \times id$ by an equivalence which is the identity when $v = 0$. Lastly, an unfolding f is $\mathcal{G}$-versal if for any other unfolding $g : \mathbb{C}^{s+r},0 \to \mathbb{C}^{t+r},0$ of f_0, there is a germ $\lambda : \mathbb{C}^r,0 \to \mathbb{C}^q,0$ such that $\lambda^* f(x,v) = (\bar{f}(x,\lambda(v)), v)$ is $\mathcal{G}$-equivalent to g.

Tangent spaces

For $\mathbb{C}^s,0$ with local coordinates x, we denote the ring of holomorphic germs $\mathcal{O}_{\mathbb{C}^s,0}$ by $\mathcal{C}_x$ with maximal ideal m_x, and similarly with u also denoting local coordinates for $\mathbb{C}^q$, $\mathcal{C}_{x,u}$ denotes $\mathcal{O}_{\mathbb{C}^{s+q},0}$, etc. Also for $f_0 : \mathbb{C}^s,0 \to \mathbb{C}^t,0$, the ring homomorphism $f_0^*: \mathcal{C}_y \to \mathcal{C}_x$, induced by composition, will be understood without being explicitly stated.

The tangent space to $\mathcal{C}_{s,t}$ at f_0 consists of germs of vector fields $\zeta : \mathbb{C}^s,0 \to T\mathbb{C}^t$ such that $\pi \circ \zeta = f_0$ and is denoted by $\theta(f_0) \cong \mathcal{C}_x\left\{\frac{\partial}{\partial y_i}\right\}$ (here the R-module generated by $\varphi_1,\ldots,\varphi_k$ is denoted by $R\{\varphi_1,\ldots,\varphi_k\}$ or $R\{\varphi_i\}$ if k is understood). Also, $\theta_s = \theta(id_{\mathbb{C}^s}) \cong \mathcal{C}_x\left\{\frac{\partial}{\partial x_i}\right\}$ and similarly for θ_t. The extended tangent spaces to $\mathcal{A}$ and $\mathcal{K}$ (which allow movement of the source and/or target) are given by

$$T\mathcal{A}_e \cdot f_0 = \mathcal{C}_x\left\{\frac{\partial f_0}{\partial x_i}\right\} + \mathcal{C}_y\left\{\frac{\partial}{\partial y_i}\right\}$$

$$T\mathcal{K}_e \cdot f_0 = \mathcal{C}_x\left\{\frac{\partial f_0}{\partial x_i}\right\} + f_0^* m_y \cdot \mathcal{C}_x\left\{\frac{\partial}{\partial y_i}\right\}.$$

For the tangent space for $\mathcal{K}_V$, we consider the module of vector fields tangent to V. If I(V) denotes the ideal of germs vanishing on V, then we let

$$\theta_V = \{\zeta \in \theta_t : \zeta(I(V)) \subseteq I(V)\}.$$

This is denoted Derlog(V) by Saito [Sa]; however, we use this simpler notation as there is no danger of confusion with other notions. θ_V extends to a sheaf of vector fields tangent to V, Θ_V which is easily seen to be coherent [Sa]. If $\{\eta_i\}_{i=1}^m$ denotes a set of generators for θ_V, then

$$T\mathcal{K}_{V,e} \cdot f_0 = \mathcal{C}_x\left\{\frac{\partial f_0}{\partial x_i}\right\} + \mathcal{C}_x\left\{\eta_i \circ f_0\right\}.$$

For $\mathcal{G} = \mathcal{A}, \mathcal{K}$ or $\mathcal{K}_V$, we denote the normal space by

$$N\mathcal{G}_e \cdot f_0 = \theta(f_0)/T\mathcal{G}_e \cdot f_0$$

and the $\mathcal{G}_e$-codimension of f_0 is $\dim_{\mathbb{C}} N\mathcal{G}_e \cdot f_0$.

The versality theorem allows us to relate several different approaches to versality (Martinet [Ma1] for $\mathcal{A}$ and $\mathcal{K}$, [D2] or [D1] for $\mathcal{K}_V$). For any unfolding $f:\mathbb{C}^{s+q},0\rightarrow\mathbb{C}^{t+q},0$

we let $\partial_j f = \frac{\partial \bar{f}}{\partial u_j}\big|_{u=0}$.

Theorem (versality theorem) : *For* $\mathcal{G}=\mathcal{A}$, $\mathcal{K}$, *or* $\mathcal{K}_V$, *and an unfolding* f *of* f_0 *the following are equivalent:*

i) f *is* $\mathcal{G}$*-versal*

ii) $T\mathcal{G}_e\cdot f_0 + \langle\partial_1 f,\ldots,\partial_q f\rangle = \theta(f_0)$

iii) *any unfolding* f_1 *of* f_0 *which extends* f *is a* $\mathcal{G}$*-trivial extension.*

Note: If $f:\mathbb{C}^{s+q},0\rightarrow\mathbb{C}^{t+q},0$ is $\mathcal{G}$-versal then $q \geq \mathcal{G}_e$-codim(f_0), and if they are equal f is said to be $\mathcal{G}$*-miniversal.*

Furthermore, f_0 is infinitesimally stable if $T\mathcal{A}_e\cdot f_0 = \theta(f_0)$; by Mather [M-IV], if f_0 has rank 0 then an unfolding f of f_0 is infinitesimally stable when viewed as a germ of a mapping if and only if

$$T\mathcal{K}_e\cdot f_0 + \langle\partial_1 f,\ldots,\partial_q f, \frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_t}\rangle = \theta(f_0)\,.$$

Hence, any f_0 with $\mathcal{K}_e$-codim$(f_0) < \infty$ has an unfolding f which is infinitesimally stable. Then, any unfolding of f is $\mathcal{A}$-equivalent to f × id.

Examples for $\mathcal{K}_V$-equivalence

Example (1.1):

Let $(V,0)\subset(\mathbb{C}^4,0)$, with coordinates (X,Y,Z,W), be defined by $YW^2-Z^2=0$. Then, V=Whitney umbrella × $\mathbb{C}$ and is parametrized by $F(x,y,u)=(x,y^2,uy,u)$. Consider $g_0:\mathbb{C}^3,0\longrightarrow\mathbb{C}^4,0$ defined by $g_0(x,y,z)=(x,y,z,p(x,y))$. It can be shown that θ_V is generated by

$$2Y\frac{\partial}{\partial Y}+Z\frac{\partial}{\partial Z},\quad Z\frac{\partial}{\partial Z}+W\frac{\partial}{\partial W},\quad WY\frac{\partial}{\partial Z}+Z\frac{\partial}{\partial W},\quad 2Z\frac{\partial}{\partial Y}+W^2\frac{\partial}{\partial Z},\quad \frac{\partial}{\partial X}\,.$$

We denote these by $\{\eta_i\}_{i=1}^5$. Since

$$\mathcal{C}_{x,y,z}\left\{\frac{\partial}{\partial X},\ldots,\frac{\partial}{\partial W}\right\}=\mathcal{C}_{x,y,z}\left\{\frac{\partial g_0}{\partial x},\frac{\partial g_0}{\partial y},\frac{\partial g_0}{\partial z},\frac{\partial}{\partial W}\right\},$$

then modulo $\mathcal{C}_{x,y,z}\left\{\frac{\partial g_0}{\partial x}, \frac{\partial g_0}{\partial y}, \frac{\partial g_0}{\partial z}\right\}$, $\frac{\partial}{\partial X}$, $\frac{\partial}{\partial Y}$, and $\frac{\partial}{\partial Z}$ are equal respectively to $-\frac{\partial p}{\partial x}\frac{\partial}{\partial W}$, $-\frac{\partial p}{\partial y}\frac{\partial}{\partial W}$, and 0. Consequently, $\eta_i \circ g_0$ for i = 1 to 5 equal, respectively,

$$-2y\frac{\partial p}{\partial y}\cdot\frac{\partial}{\partial W},\quad p(x,y)\frac{\partial}{\partial W},\quad z\frac{\partial}{\partial W},\quad -2z\frac{\partial p}{\partial y}\cdot\frac{\partial}{\partial W},\quad -\frac{\partial p}{\partial x}\cdot\frac{\partial}{\partial W}.$$

Hence,

$$N\mathcal{K}_{V,e}\cdot g_0 = \mathcal{C}_{x,y,z}\left\{\frac{\partial}{\partial X},\dots,\frac{\partial}{\partial W}\right\} / \mathcal{C}_{x,y,z}\left\{\frac{\partial g_0}{\partial x},\dots,\frac{\partial g_0}{\partial z}, \eta_i \circ g_0\right\}$$

$$\xrightarrow{\sim} \mathcal{C}_{x,y,z}\left\{\frac{\partial}{\partial W}\right\} / \left\{y\frac{\partial p}{\partial y}, p, z, \frac{\partial p}{\partial x}\right\}\cdot\frac{\partial}{\partial W}$$

$$\xrightarrow{\sim} \mathcal{C}_{x,y} / \left(y\frac{\partial p}{\partial y}, p, \frac{\partial p}{\partial x}\right). \tag{1.2}$$

If we pull back F via g_0 to form $f_0(x,y) = (x,y^2,yp(x,y^2))$

$$\begin{array}{ccc} \mathbb{C}^3,0 & \xrightarrow{F} & \mathbb{C}^4,0 \\ \uparrow & & \uparrow g_0 \\ \mathbb{C}^2,0 & \xrightarrow{f_0} & \mathbb{C}^3,0 \end{array}$$

then Mond computes $N\mathcal{A}_e\cdot f_0$ be exactly (1.2) [Mo1].

Example (1.3)

Let $f_0 : \mathbb{C}^n,0 \to \mathbb{C},0$ be a weighted homogeneous germ defining an isolated singularity. Also, let $F : \mathbb{C}^{n+q},0 \to \mathbb{C}^{1+q},0$ be its versal unfolding, with V=discriminant of F. Then, Saito [Sa] gives the following construction for the generators of θ_V. Let $\{\varphi_i\}_{i=1}^q$ be a basis for $N\mathcal{A}_e\cdot f_0$ and let $\varphi_0 = 1$. We may assume up to equivalence that F is given by (

$\bar{F}(x,u), u) = (f_0(x) + \sum_{i=1}^{q} u_i\varphi_i, u)$.

$$\bar{F}\cdot\varphi_i = \sum_{j=0}^{q} a_{ij}(u)\varphi_j \mod \left(\frac{\partial\bar{F}}{\partial x_1},\dots,\frac{\partial\bar{F}}{\partial x_n}\right)$$

Let

$$\eta_i = -y\cdot\frac{\partial}{\partial u_i} + \sum_{j=1}^{q} a_{ij}\frac{\partial}{\partial u_j} + a_{i0}\frac{\partial}{\partial y} \quad \text{for } i > 0$$

and

$$\eta_0 \ = \ \frac{1}{d}\cdot\text{Euler vector field} \qquad (d = wt(f_0)) .$$

Then, $\{\eta_i\}_{i=0}^{q}$ generate θ_V. Let $g_0 : \mathbb{C} \longrightarrow \mathbb{C}^{1+q}$ be defined by $g_0(y) = (y, 0)$. Then,

$\eta_i \circ g_0 = -y\frac{\partial}{\partial u_i}$ for $i > 0$ or $= y\frac{\partial}{\partial y}$ for $i = 0$. Thus,

$$N\mathcal{K}_{V,e}\cdot g_0 \ = \ \mathcal{C}_y\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial u_i}\right\} / \ \mathcal{C}_y\left\{\frac{\partial g_0}{\partial y}, \eta_i \circ g_0\right\}$$

$$\tilde{\to} \ \mathcal{C}_y \ / \ m_y\left\{\frac{\partial}{\partial u_i}\right\}$$

$$\tilde{\to} \bigoplus_{i=1}^{q} \mathbb{C} \qquad \text{(here } q = \tau(f_0) - 1)$$

Again g_0 pulls back F to give $f_0 : \mathbb{C}^n,0 \to \mathbb{C},0$.

$$N\mathcal{A}_e\cdot f_0 \ \tilde{\to} \ \mathcal{C}_x / \left(\frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n}\right) + \langle 1 \rangle .$$

Since f_0 is weighted homogeneous, $f_0 \in \left(\frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n}\right)$. Thus, as a $\mathcal{C}_y$-module.

$$N\mathcal{A}_e\cdot f_0 \ \tilde{\to} \ \bigoplus_{i=1}^{\mu-1} \mathbb{C} .$$

Since $\mu = \tau$, these $\mathcal{C}_y$-modules are isomorphic.

Infinitesimal Conditions for Triviality

Next, the relations we shall establish between $\mathcal{A}$ and $\mathcal{K}_V$-equivalence are most easily established at the infinitesimal level. For this reason, we recall the infinitesimal conditions for triviality.

Let $f : \mathbb{C}^{s+q}, 0 \to \mathbb{C}^{t+q}, 0$ be an unfolding of f_0 and let $f_1 : \mathbb{C}^{s+q+r}, 0 \to \mathbb{C}^{t+q+r}, 0$ extend f (with local coordinates u for $\mathbb{C}^q$ and v for $\mathbb{C}^r$).

Criterion for $\mathcal{A}$-triviality: f_1 is an $\mathcal{A}$-trivial extension of f if and only if there exist vector fields $\xi_i \in \mathcal{C}_{x,u,v}\left\{\frac{\partial}{\partial x_i}\right\}$, $\delta_i \in \mathcal{C}_{y,u,v}\left\{\frac{\partial}{\partial y_i}\right\}$ and $\zeta_i \in \mathcal{C}_{u,v}\left\{\frac{\partial}{\partial u_i}\right\}$ such that

$$(1.4) \qquad \frac{\partial \bar{f}_1}{\partial v_i} = -\xi_i(\bar{f}_1) - \zeta_i(\bar{f}_1) + \delta_i \circ f_1 \qquad 1 \leq i \leq r .$$

Also, if $q = 0$ and $f_1 = f_0$, then f_1 is an $\mathcal{A}$-trivial unfolding of f_0 if and only if (1.4) can be solved with $\zeta_i \equiv 0$. Furthermore, in this case f_1 is an $\mathcal{A}$-trivial family if and only if we can choose $\xi_i \in m_x \cdot \mathcal{C}_{x,v}\left\{\frac{\partial}{\partial x_i}\right\}$, $\delta_i \in m_y \cdot \mathcal{C}_{y,v}\left\{\frac{\partial}{\partial v_i}\right\}$ (and again $\zeta_i \equiv 0$).

This criterion follows from the reduction lemma for $\mathcal{A}$-equivalence in Martinet [Ma1]. The converse follows by differentiating the trivialization with respect to coordinates trivializing the unfoldings in the v_i-directions.

Criterion for $\mathcal{K}_V$-triviality: f_1 is a $\mathcal{K}_V$-trivial extension of f if and only if there are vector fields $\xi_i \in \mathcal{C}_{x,u,v}\left\{\frac{\partial}{\partial x_i}\right\}$, $\delta_i \in \mathcal{C}_{y,u,v}\{\eta_i\}$ (where $\{\eta_i\}$ generate θ_V) and $\zeta_i \in \mathcal{C}_{u,v}\left\{\frac{\partial}{\partial u_i}\right\}$ such that (1.4) is satisfied.

Similarly, if $q = 0$, f_1 is a $\mathcal{K}_V$-trivial unfolding of f_0 if (1.4) can be solved with $\zeta_i \equiv 0$, or a $\mathcal{K}_V$-trivial family if (1.4) can be solved with $\xi_i \in m_x \mathcal{C}_{x,v}\left\{\frac{\partial}{\partial x_i}\right\}$. This follows for $\mathcal{K}_V$-equivalence by the corresponding reduction lemmas in [D1] or [D2].

Geometric Criteria for Finite Determinancy

Finite $\mathcal{A}$-determinacy and finite $\mathcal{K}_V$-determinacy each have geometric characterizations. For $\mathcal{G} = \mathcal{A}$ and $\mathcal{K}$, by Mather [M-III], and $\mathcal{K}_V$, by [D2], finite $\mathcal{G}$-determinacy of f_0 is equivalent to finite $\mathcal{G}$-codimension of f_0. Via this, there is the geometric characterization of finite $\mathcal{A}$-determinacy by Gaffney and Mather: f_0 is finitely $\mathcal{A}$-determined if and only if f_0 is infinitesimally stable in a punctured neighbourhood of 0, i.e. there is a representative of f, $f_1 : U \to \mathbb{C}^t$ such that f_1 is infinitesimally stable on $U \backslash \{0\}$.

For finite $\mathcal{K}_V$-determinacy, let $\{\eta_i\}_{i=1}^m$ be a set of generators of θ_V. By coherence they also generate $\Theta_{V,y}$ in a neighbourhood of 0. By $f_0 : \mathbb{C}^s,0 \to \mathbb{C}^t,0$ *being transverse to* $(V,0)$ *at* x_0 we shall mean

$$df_0(x_0)(T\mathbb{C}^s) + \langle \eta_{1(f(x_0))},\dots,\eta_{m(f(x_0))}\rangle = T\mathbb{C}^t_{f(x_0)}.$$

Then, f_0 is finitely $\mathcal{K}_V$-determined if and only if f_0 is transverse to V in a punctured neighbourhood of 0 (although this characterization was stated in [D2] for finite map germs f_0, the proof given there works in general).

$\mathcal{K}_V$-equivalence and suspension

Lastly, we relate $\mathcal{K}_V$-equivalence to $\mathcal{K}_{V'}$-equivalence for $V' = V \times \mathbb{C}^r$. Given $f_0 : \mathbb{C}^s,0 \to \mathbb{C}^t,0$ and $g : \mathbb{C}^t,0 \to \mathbb{C}^p,0$ we let $g_* f_0 = g \circ f_0$. For an unfolding $f : \mathbb{C}^{s+q},0 \to \mathbb{C}^{t+q},0$ of f_0, we define $g_* f(x,u) = (g \circ \bar{f}(x,u), u)$ which is an unfolding of $g_* f_0$. We consider $i : \mathbb{C}^t,0 \to \mathbb{C}^{t+r},0$ with $i(y) = (y, 0)$ and $\pi : \mathbb{C}^{t+r},0 \to \mathbb{C}^t,0$ with $\pi(y,w) = y$. We also note g induces a $\mathcal{C}_x$-module homomorphism $g_*:\theta(f_0) \to \theta(g_* f_0)$, defined by $g_*(\zeta) = dg(\zeta)$.

We say that $V,0 \subset \mathbb{C}^t,0$ and $V_1,0 \subset \mathbb{C}^p,0$ are g-*related* if for a set of generators $\{\eta_i\}_{i=1}^m$ of θ_V there are $\eta'_i \in \theta_{V_1}$ so that $g_*(\eta_i) = \eta'_i \circ g$. For example, $V,0 \subset \mathbb{C}^t,0$ and $V' = V \times \mathbb{C}^r,0 \subset \mathbb{C}^{t+r}$ are both i and π related.

Proposition 1.5: *With the preceding notation, let* f *be an unfolding of* f_0 *and* f_1 *an extension of* f.

i) *Suppose* $V,0 \subset \mathbb{C}^t,0$ *and* $V_1,0 \subset \mathbb{C}^p,0$ *are* g-*related; if* f *is a* $\mathcal{K}_V$-*trivial unfolding (respectively family) then* $g_* f$ *is a* $\mathcal{K}_{V_1}$-*trivial unfolding (respectively family); also if* f_1 *is a* $\mathcal{K}_V$-*trivial extension of* f *then* $g_* f_1$ *is a* $\mathcal{K}_{V_1}$-*trivial extension of* $g_* f$.

ii) i_* *and* π_* *induce isomorphisms of* $\mathcal{C}_y$ *(respectively* $\mathcal{C}_{y,w}$*)-modules)*

$$i_* : N\mathcal{K}_{V,e} \cdot f_0 \xrightarrow{\sim} N\mathcal{K}_{V',e} i_* f_0 \quad \text{and} \quad \pi_* : N\mathcal{K}_{V',e} f'_0 \xrightarrow{\sim} N\mathcal{K}_{V,e} \pi_* f'_0$$

iii) f *is* $\mathcal{K}_V$-*versal if and only if* $i_* f$ *is* $\mathcal{K}_{V'}$-*versal,*
f′ *is* $\mathcal{K}_{V'}$-*versal if and only if* $\pi_* f$ *is* $\mathcal{K}_V$-*versal.*

Proof: i) By the infinitesimal criterion we may solve

$$\frac{\partial \bar{f}_1}{\partial v_i} = - \xi_i(\bar{f}_1) - \zeta_i(\bar{f}_1) + \delta_i \circ f_1 .$$

Applying dg, we obtain

$$\textbf{(1.6)} \qquad \frac{\partial(g \circ \bar{f}_1)}{\partial v_i} = - \xi_i (g \circ \bar{f}_1) - \zeta_i (g \circ \bar{f}_1) + dg\,(\delta_i \circ f_1)$$

If $\delta_i = \sum h_{ij}\eta_j$ with $h_{ij} \in \mathcal{C}_{x,u,v}$, then

$$\begin{aligned} dg\,(\delta_i) \circ f_1 &= \sum h_{ij}(dg(\eta_i) \circ f_1) = \sum h_{ij} \cdot \eta'_i \circ g \circ f_1 \\ &= \eta^{(i)'} \circ (g \circ \bar{f}) \quad \text{with} \quad \eta^{(i)'} = \sum h_{ij}\eta'_i . \end{aligned}$$

Substituting into (1.6) satisfies the criterion for triviality for $g_* f_1$. The cases of triviality of unfoldings or families are similar.

ii) Suppose V and V_1 are g-related:

If $\zeta \in T\mathcal{K}_{V,e} \cdot f_0$, then $\zeta = \frac{\partial f}{\partial t}\big|_{t=0}$ for f a 1-parameter $\mathcal{K}_V$-trivial unfolding of f_0.

Then, $g_*(\zeta) = \frac{\partial(g \circ \bar{f})}{\partial t}\big|_{t=0} \in T\mathcal{K}_{V_1,e} g_* f_0$. Thus, g_* induces a map

$$g_* : N\mathcal{K}_{V,e} \cdot f_0 \rightarrow N\mathcal{K}_{V_1,e} \cdot g_* f_0 .$$

It remains to show that this is an isomorphism for $g = i$ and $g = \pi$. However, by naturality $\pi_* \circ i_* = (\pi \circ i)_* = id_* = id$. If we can show π_* is an isomorphism on normal spaces then so is i_*. Explicitly if $f'_0 : \mathbb{C}^s, 0 \rightarrow \mathbb{C}^{t+r}, 0$ has components $f'_0 = (f'_{0,1}, f'_{0,2})$ then $\theta_{V'}$ is generated by $\{\eta_i\} \cup \left\{\frac{\partial}{\partial w_j}\right\}$ where $\{\eta_i\}$ are a set of generators for θ_V; hence,

$$\begin{aligned} N\mathcal{K}_{V',e} \cdot f'_0 &= \mathcal{C}_X\left\{\frac{\partial}{\partial y_i}, \frac{\partial}{\partial w_j}\right\} / \left(\mathcal{C}_X\left\{\frac{\partial f_0}{\partial x_i}\right\} + \mathcal{C}_X\left\{\eta_i \circ f'_0, \frac{\partial}{\partial w_j}\right\}\right) \\ &\tilde{\rightarrow} \ \mathcal{C}_X\left\{\frac{\partial}{\partial y_i}\right\} / \mathcal{C}_X\left\{\frac{\partial f'_{01}}{\partial x_i}\right\} + \mathcal{C}_X\left\{\eta_i \circ f'_{01}\right\} \\ &\tilde{\rightarrow} \ N\mathcal{K}_{V,e} \cdot \pi_* f'_0 \end{aligned}$$

and the projection is exactly π_*.

iii) Finally since π_* and i_* commute with $\frac{\partial}{\partial u_i}$, condition ii) of the versality theorem yields the results. □

§2. Relating $\mathcal{A}$ and $\mathcal{K}_V$-equivalence

In this section we deduce relations between $\mathcal{K}_V$-equivalence of unfoldings and families and $\mathcal{A}$-equivalence for the corresponding unfoldings and families induced via pullback. As a consequence we obtain the numerical equality between $\mathcal{A}_e$-codimension and $\mathcal{K}_{V,e}$-codimension described in the introduction.

Because the $\mathcal{A}$-equivalence and $\mathcal{K}_V$-equivalence are for germs which map between different spaces, we slightly change notation from the preceding section. Consider a germ $f_0 : \mathbb{C}^n,0 \to \mathbb{C}^p,0$ which has finite $\mathcal{K}$-codimension. As mentioned in the preceding section, there is an unfolding $F : \mathbb{C}^{n'},0 \to \mathbb{C}^{p'},0$ of f_0 which is stable when viewed as a germ. We shall refer to such an unfolding as a *stable unfolding* of f_0. There is an inclusion $g_0 : \mathbb{C}^p,0 \to \mathbb{C}^{p'},0$ given by $g_0(y) = (y,0)$ and g_0 is transverse to F, and f_0 may be viewed as being obtained by the fiber product, i.e. pull-back of F by g_0.

$$\begin{array}{ccc} \mathbb{C}^{n'},0 & \xrightarrow{F} & \mathbb{C}^{p'},0 \\ \uparrow & & \uparrow g_0 \\ \mathbb{C}^n,0 & \xrightarrow{f_0} & \mathbb{C}^p,0 \end{array}$$

Also, given an unfolding g of g_0 we have an induced unfolding f of f_0 obtained as the fiber product of F and $\bar{g}$. We shall relate the $\mathcal{A}$-equivalence of f_0 and its unfoldings with the $\mathcal{K}_V$-equivalence of g_0 and its unfoldings.

By [M2], we may choose a representative of F, again denoted by $F : U \to W$ such that if $\Sigma(F) = \{x \in U : \text{rk}\, df(x) < p'\}$ denotes the critical set of F, then

1) $F^{-1}(0) \cap \Sigma(F) = \{0\}$
2) $F \mid \Sigma(F)$ is finite to one
3) F is stable.

We let $D(F) = F(\Sigma(F))$. If $n \geq p$ this is the discriminant of F, while if $n < p$ it is the image of F. We denote D(F) by V.

Remark 2.1: Any unfolding of F is $\mathcal{A}$-equivalent to $F \times \mathrm{id}$. If we were to replace F by $F \times \mathrm{id}_{\mathbb{C}^r}$ then $D(F \times \mathrm{id}_{\mathbb{C}^r}) = D(F) \times \mathbb{C}^r = V'$, say. By proposition 1.5, $\mathcal{K}_V$-equivalence for g_0 and its unfoldings is equivalent to $\mathcal{K}_{V'}$-equivalence for $i_* g_0 = i \circ g_0 : \mathbb{C}^p,0 \longrightarrow \mathbb{C}^{p'},0 \hookrightarrow \mathbb{C}^{p'+r},0$. Thus, it does not matter which stable unfolding of f_0 we choose.

A principal reason for the close relation between $\mathcal{A}$ and $\mathcal{K}_V$-equivalence is the characterization of θ_V due to Arnold [A] and Saito [Sa] (see also Bruce [Br] and Terao [T]).

Lemma 2.2: *With the preceding notation,*

$$\theta_V = \{\eta \in \theta_{p'}: \text{ there is a } \xi \in \theta_{n'} \text{ so that } \xi(F) = \eta \circ F\},$$

that is, the set of liftable vector fields.

Proof: The proofs for $n \geq p$ are given in the above references. The argument for $n < p$ is the same; by Hartogs' theorem η lifts if and only if it lifts off a set of codimension 2 in $\mathbb{C}^{n'}$. As F is stable, the only singular points of codimension 1 occur at double points when $p = n + 1$. Clearly η lifts from the regular points of V. At double points, F is a suspension of the germ $\mathbb{C},0 \perp\!\!\perp \mathbb{C},0 \to \mathbb{C}^2,0$ defined by $x \mapsto x$, $y \mapsto y$ in $\mathbb{C}^2$ with image $x \cdot y = 0$. The vector fields tangent to this set are generated by $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$ and clearly lift. The converse is immediate since $dF(\xi)$ is tangent to V_{reg} so for any $h \in I(V)$, $\xi(h) = 0$ on V_{reg} and hence by continuity on V. □

The first question to resolve is the relation between g_0 being finitely $\mathcal{K}_V$-determined and f_0 being finitely $\mathcal{A}$-determined.

Proposition 2.3 *f_0 is finitely $\mathcal{A}$-determined if and only if g_0 is finitely $\mathcal{K}_V$-determined.*

Proof: For both directions we use the geometric criterion from the preceding section.

⇐ As g_0 is finitely $\mathcal{K}_V$-determined it is transverse to V in a punctured neighbourhood of 0. Let W be such a punctured neighbourhood with a representative of g_0 still denoted by g_0. Let $\{\eta_i\}$ be a set of vector fields in θ_V which generate $\Theta_{V,y'}$ for y' in a neighbourhood of 0 which includes W (by shrinking W if necessary). For $y \in W$, let $S = F^{-1}(g_0(y)) \cap \Sigma(F)$, which is finite. For each i let ξ_i be a lift of η_i which, by shrinking U if necessary, is defined on U. Then, $F : \mathbb{C}^{n'},S \to \mathbb{C}^{p'},g_0(y)$ is stable. Pick a subset $\{\eta_1,\dots,\eta_r\}$ of the above set $\{\eta_i\}$ such that $\langle \eta_{1(g_0(y))},\dots,\eta_{r(g_0(y))} \rangle$ spans a complementary subspace to

$dg_0(y)(T\mathbb{C}^p)$. Then, since $\xi_i(F) = F \circ \eta_i$, by a standard argument in e.g. Martinet [Ma1], $F : \mathbb{C}^{n'},S \to \mathbb{C}^{p'},g_0(y)$ is $\mathcal{A}$-equivalent as a multi-germ to $f_0 \times id : \mathbb{C}^{n'},S \to \mathbb{C}^{p'},y$. This implies that $f_0 : \mathbb{C}^n,S \to \mathbb{C}^p,y$ is stable (f_0 is stable if and only if $f_0 \times id$ is by the infinitesimal criteria of Mather [M-IV]). As y was an arbitrary point of W, f_0 is stable in a punctured neighbourhood of 0 and so is finitely $\mathcal{A}$-determined.

Conversely, if f_0 is finitely $\mathcal{A}$-determined then for y in a punctured neighbourhood W of 0, $f_0 : \mathbb{C}^n,S \to \mathbb{C}^p,y$ is stable. Hence, $F : \mathbb{C}^{n'},S \to \mathbb{C}^{p'},y$ is an $\mathcal{A}$-trivial unfolding of f_0. Thus, there are vector fields ξ_i, η'_i defined near S and y so that $\xi_i(F) = \eta'_i \circ F$ and $\{\eta'_{i(y)}\}$ span a subspace complementary to $\mathbb{C}^p$. Thus, $\eta'_i \in \Theta_{V,y}$. By choosing W smaller if necessary, $\{\eta_i\}$ generate $\Theta_{V,y}$ for $y \in W$. Hence, the subspace spanned by $\{\eta'_{i(y)}\}$ is contained in that spanned by $\{\eta_{i(y)}\}$. Thus, $\mathbb{C}^p$ is transverse to V at y. Thus, $\mathbb{C}^p$ is transverse to V in the punctured neighbourhood W, i.e. g_0 is transverse to V on W and hence is finitely $\mathcal{K}_V$-determined. □

Second, we relate $\mathcal{K}_V$-triviality of unfoldings of g_0 with $\mathcal{A}$-triviality of unfoldings of f_0. We let $g(x,u)$ be an unfolding of g_0 and $g_1(x,u,v) : \mathbb{C}^{p+q+r},0 \to \mathbb{C}^{p'+q+r},0$ an extension of g. We let f and f_1 denote the induced unfoldings of f_0.

Proposition 2.4: i) *If* g *is a* $\mathcal{K}_V$*-trivial unfolding (respectively* $\mathcal{K}_V$*-trivial family) then* f *is an* $\mathcal{A}$*-trivial unfolding (respectively* $\mathcal{A}$*-trivial family).*
ii) *If* g_1 *is a* $\mathcal{K}_V$*-trivial extension of* g *then* f_1 *is an* $\mathcal{A}$*-trivial extension of f.*

Proof: We give the proof of ii); that of i) is analogous (and slightly easier).

By the infinitesimal criterion, there exist germs of vector fields $\zeta_i \in C_{y,u,v}\left\{\frac{\partial}{\partial y_i}\right\}$, $\chi_i \in C_{y,u,v}\{\eta_i\}$, and $\gamma_i \in C_{u,v}\left\{\frac{\partial}{\partial u_i}\right\}$ (with $\{\eta_i\}$ generating θ_V) such that

$$(2.5) \qquad \frac{\partial \bar{g}_1}{\partial v_i} = -\zeta_i(\bar{g}_1) - \gamma_i(\bar{g}_1) + \chi_i \circ g_1 .$$

Let $\{\xi_i\}$ denote the lifts of the $\{\eta_i\}$. If $\chi_i = \sum h_{ij}\eta_j$, let $\delta_i = \sum h'_{ij}\xi_j$ with $h'_{ij} = h''_{ij} \circ (F \times \mathrm{id})$. To define h''_{ij}, we note that h_{ij} is a germ defined on $\mathbb{C}^{p+q+r}$; however, $g_1 : \mathbb{C}^{p+q+r},0 \longrightarrow \mathbb{C}^{p'+q+r},0$ is a germ of an immersion. Thus, $h_{ij} = g_1^* h''_{ij}$ for some h''_{ij} on $\mathbb{C}^{p'+q+r},0$. We also replace χ_i by $\chi'_i = \sum h''_{ij}\eta_j$ where η_j also denotes its trivial extension to $\mathbb{C}^{p'+q+r},0$. Then (2.5) remains valid if we replace χ_i by χ'_i since $\chi'_i \circ g_1 = \chi_i \circ g_1$. Also

(2.6) $$\delta_i\,(F \times \mathrm{id}) = \chi'_i \circ (F \times \mathrm{id})\,.$$

Now, f_1 is formed from g_1 and $F \times \mathrm{id}$ by fiber product. We make this explicit. Let

$$H_1 : \mathbb{C}^{n'+p+q+r},0 \longrightarrow \mathbb{C}^{2p'+q+r},0$$

be defined by $H_1(x',y,u,v) = (\bar{F}(x'), \bar{g}_1(y,u,v), u,v)$, and

$$H : \mathbb{C}^{n'+p+q},0 \longrightarrow \mathbb{C}^{2p'+q},0$$

by $H(x',y,u) = (F(x'), \bar{g}(y,u), u)$. Let,

$$\Delta_1 = \{(y',y',u,v) : y' \in \mathbb{C}^{p'}\},$$

$$\Delta = \{(y',y',u) : y' \in \mathbb{C}^{p'}\}\,.$$

Then, f_1 and f are the restrictions of H_1 and H

$$H_1 : \ H_1^{-1}(\Delta_1) \longrightarrow \Delta_1 \qquad\qquad H : \ H^{-1}(\Delta) \longrightarrow \Delta\,.$$

We wish to prove that $H_1 \mid H_1^{-1}(\Delta_1)$ is an $\mathcal{A}$-trivial extension of $H \mid H^{-1}(\Delta)$.

Weclaim

(2.7) $$\frac{\partial \bar{H}_1}{\partial v_i} = \left(0, \frac{\partial \bar{g}_1}{\partial v_i}\right) = -(\delta_i,\zeta_i)\,H_1 - \gamma_i(H_1) + (\chi'_i,\chi'_i) \circ H_1$$

for on the first component

$$0 = -\delta_i(F \times \mathrm{id}) - 0 + \chi'_i \circ (F \times \mathrm{id}),$$

and on the second component

$$\frac{\partial \bar{g}_1}{\partial v_i} = -\zeta_i(\bar{g}_1) - \gamma_i(\bar{g}_1) + \chi'_i \circ g_1$$

which follow by (2.5) and (2.6). Also,

$$\tilde{\eta}_i = \frac{\partial}{\partial v_i} + \gamma_i + (\chi'_i,\chi'_i)$$

is tangent to Δ_1; and if we let

$$\tilde{\xi}_i = \frac{\partial}{\partial v_i} + \gamma_i + (\delta_i, \zeta_i)$$

then

$$\tilde{\xi}_i (H_1) = H_1 \circ \tilde{\eta}_i .$$

Thus, $\tilde{\xi}_i$ is tangent to $H_1^{-1}(\Delta_1)$. Then, the restrictions $\tilde{\eta}_i \mid \Delta$ and $\tilde{\xi}_i \mid H_1^{-1}(\Delta_1)$ give the vector fields which provide the infinitesimal trivialization of $H_1 \mid H_1^{-1}(\Delta_1)$ as an extension of $H \mid H^{-1}(\Delta)$. □

Now we are in a position to establish the equality of codimensions before we even define the algebraic homomorphism between normal spaces. It is enough to show: 1) if g is a $\mathcal{K}_V$-versal unfolding of g_0 then the induced f is an $\mathcal{A}$-versal unfolding of g and 2) there is an $\mathcal{A}$-miniversal unfolding f of f_0 induced by an unfolding g with g $\mathcal{K}_V$-versal. For by the versality theorem, 1) implies $\mathcal{A}_e$-codim$(f_0) \leq \mathcal{K}_{V,e}$-codim$(g_0)$ while 2) implies the reverseinequality.

Then, 1) is established by

Lemma 2.8: *Let* g *be a* $\mathcal{K}_V$*-versal unfolding of* g_0*, then* f *is an* $\mathcal{A}$*-versal unfolding of* f_0.

Proof: Let f_1 be an extension of f. To prove that f is $\mathcal{A}$-versal, it is sufficient to prove that any such f_1 is an $\mathcal{A}$-trivial extension of f. If we can show that f_1 is induced by a g_1 which is an extension of g, then, by the $\mathcal{K}_V$-versality of g, g_1 is a $\mathcal{K}_V$-trivial extension of g; and by proposition 1.5, f_1 is an $\mathcal{A}$-trivial extension of f. We actually prove a variant of this where g_0, g and g_1 are replaced by related germs h_0, h and h_1, which induce f_0, f and f_1 from a larger stable unfolding so we can still apply proposition 1.5.

To define the h's, we enlarge the stable unfolding F to include explicitly all of the unfoldings under consideration. We represent F as an unfolding $F(x,w) = (\bar{F}(x,w), w)$. The unfolding $g(y,u) = (\bar{g}(y,u),u)$, $\bar{g}: \mathbb{C}^{p+q},0 \longrightarrow \mathbb{C}^{p'},0$ has the form $(y,w) = \bar{g}(y,u) = (\bar{g}'(y,u), \bar{g}''(y,u))$. Define a map $\varphi : \mathbb{C}^{p+q},0 \longrightarrow \mathbb{C}^{p'+q},0$ by $\varphi(y,w,u) = (\bar{g}'(y,u), \bar{g}''(y,u) + w, u)$. It is easily checked that φ is a germ of a diffeomorphism, so that $F \times \mathrm{id}$ pulls back via φ to an unfolding

$$F_1(x,u,w) = (\bar{F}_1(x,u,w), u, w)$$

and that

$$\bar{F}_1(x,u,0) = \bar{f}(x,u) \qquad \text{and} \qquad \bar{F}_1(x,0,w) = \bar{F}(x,w).$$

Consider the unfolding

$$F_2(x,u,w,v) = (\bar{F}_1(x,u,w) - \bar{f}(x,u) + \bar{f}_1(x,u,v), u,w,v).$$

Then

(2.9) $F_2(x,u,0,v) = (\bar{f}_1(x,u,v), u,0,v)$ and $F_2(x,0,w,0) = (\bar{F}(x,w), 0,w,0)$

Since F is stable, by (2.9) and the infinitesimal criterion of Mather, F_2 is stable. Then, we define $h_0: \mathbb{C}^p,0 \longrightarrow \mathbb{C}^{p'+q+r},0$, $\bar{h}: \mathbb{C}^{p+q},0 \longrightarrow \mathbb{C}^{p'+q+r},0$, and $\bar{h}_1: \mathbb{C}^{p+q+r},0 \longrightarrow \mathbb{C}^{p'+q+r},0$ by $h_0(y) = (y,0,0,0)$, $\bar{h}(y,u) = (y,u,0,0)$, and $\bar{h}_1(y,u,v) = (y,u,0,v)$. By (2.9) we see that $\bar{h}$ pulls back F_2 to give f, $\bar{h}_1$ pulls back F_2 to give f_1 and h_1 is an extension of h. If we knew that h_1 were a $\mathcal{K}_{V''}$-trivial extension of h, where $V'' = D(F_2)$, then by proposition 1.5 we could draw the desired conclusion.

To see that it is, we define $G_0: \mathbb{C}^p,0 \longrightarrow \mathbb{C}^{p'+q+r},0$ by $G_0(y) = (y, 0, 0)$ and the unfolding $G(y,u) = (\bar{G}(y,u), u)$ by $\bar{G}(y,u) = (\bar{g}(y,u),u,0)$. Then, $g_0 = \pi_* G_0$ and $\bar{g} = \pi_* \bar{G}$ for $\pi: \mathbb{C}^{p'+q+r},0 \longrightarrow \mathbb{C}^{p'},0$ the projection. Thus, by proposition 1.5, G is a $\mathcal{K}_{V'}$-versal unfolding of G_0 where $V' = V \times \mathbb{C}^{q+r}$. Also, $(\varphi \times id)_* h = G$, $(\varphi \times id)_* h_0 = G_0$, and $\varphi \times id(V'') = D(F \times id) = V'$. Since $\varphi \times id$ is a diffeomorphism, h is a $\mathcal{K}_{V''}$-versal unfolding of h_0 if and only if G is a $\mathcal{K}_{V'}$-versal unfolding of G_0, which it is. Hence, h_1 is a $\mathcal{K}_{V''}$-trivial extension of h ; and thus, f_1 is an $\mathcal{A}$-trivial extension of f. □

For 2) we let $f(x,u) = (\bar{f}(x,u), u)$ denote an $\mathcal{A}$-versal unfolding of f_0 with $f : \mathbb{C}^{p+q},0 \to \mathbb{C}^{n+q},0$. We define an unfolding of $g_0(y) = y$ by $\bar{g}(y,u) = (y,u)$.

Lemma 2.10: g *is a* $\mathcal{K}_V$*-versal unfolding of* g_0, *where* $V = D(f)$.

Proof: Since

$$T\mathcal{K}_e \cdot f_0 + \langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p} \rangle \supseteq T\mathcal{A}_e \cdot f_0$$

it follows that the $\mathcal{A}$-versal unfolding f is also a stable unfolding of f_0. Then, we may use f for our stable unfolding F.

Let g_1 be an extension of g, with additional parameters $v \in \mathbb{C}^r$. Define $\varphi : \mathbb{C}^{p+q+r},0 \longrightarrow \mathbb{C}^{p+q+r},0$ by $\varphi(y,u,v) = (\bar{g}_1(y,u,v), v)$. As g_1 is an extension of g, $\bar{g}_1(y,u,0) = (y,u)$. Hence, φ is a germ of a diffeomorphism by the inverse function theorem. We may pull back $f \times id$ by φ to obtain an unfolding $f_1 : \mathbb{C}^{n+q+r},0 \longrightarrow \mathbb{C}^{p+q+r},0$. Since $\varphi(y,u,0) = (y,u,0)$, $\bar{f}_1(x,u,0) = \bar{f}(x,u)$. Note even though f_1 is a pull-back of a trivial unfolding $f \times id$, the pull-back is not in the usual sense of unfoldings; hence, the unfolding need not be an $\mathcal{A}$-trivial extension f. However, f_1 is an extension of the unfolding f which is $\mathcal{A}$-versal. Hence, f_1 is an $\mathcal{A}$-trivial extension of f by the versality theorem.

By the infinitesimal criterion, there exist vector fields of the form

$$\chi_i = \frac{\partial}{\partial v_i} + \gamma_i + \zeta_i \qquad \delta_i = \frac{\partial}{\partial v_i} + \xi_i + \zeta_i \qquad 1 \le i \le r$$

where $\gamma_i \in \mathcal{C}_{y,u,v}\left\{\frac{\partial}{\partial y_i}\right\}$, $\xi_i \in \mathcal{C}_{x,u,v}\left\{\frac{\partial}{\partial x_j}\right\}$, $\zeta_i \in \mathcal{C}_{u,v}\left\{\frac{\partial}{\partial u_j}\right\}$ and such that

$$\delta_i(f_1) = \chi_i \circ f_1 .$$

Thus, χ_i is f_1-liftable and

$$\frac{\partial}{\partial v_i} = \chi_i - \gamma_i - \zeta_i .$$

Consider the unfoldings h and h_1 of $h_0(y) = (y,0,0)$ with $\bar{h}(y,u) = (y,u,0)$ and $\bar{h}_1(y,u,v) = (y,u,v)$.

$$\chi_i \circ \bar{h}_1 = \chi_i , \quad \zeta_i(\bar{h}_1) = \zeta_i , \quad \gamma_i(\bar{h}_1) = \gamma_i , \quad \text{and} \quad \frac{\partial \bar{h}_1}{\partial v_i} = \frac{\partial}{\partial v_i} .$$

Hence,

$$\frac{\partial \bar{h}_1}{\partial v_i} = -\gamma_i(\bar{h}_1) - \zeta_i(\bar{h}_1) + \chi_i \circ \bar{h}_1 \qquad 1 \le i \le r .$$

Hence, h_1 is a $\mathcal{K}_{V'}$-trivial extension of h where $V' = D(f_1)$.

Now, $\varphi(D(f_1)) = D(f) \times \mathbb{C}^r = V \times \mathbb{C}^r$ and $\varphi(y,u,0) = (y,u,0)$. Thus, $\varphi_* h_1$ is a $\mathcal{K}_{V\times\mathbb{C}^r}$ trivial extension of $\varphi_* h$ by proposition 1.5 and hence $g_1 = \pi_* \varphi_* h_1$ is a $\mathcal{K}_V$-trivial extension of $g = \pi_* \varphi_* h$. As g_1 was an arbitrary extension of g, g is $\mathcal{K}_V$-versal. □

Now, if g is a $\mathcal{K}_V$-miniversal unfolding of g_0 on q parameters, then the induced f is an $\mathcal{A}$-versal unfolding of f_0 by lemma 2.8. Thus, by the versality theorem, $\mathcal{K}_{V,e}$-codim $(g_0) = q \ge \mathcal{A}_e$-codim (f_0). On the other hand, if f is an $\mathcal{A}_e$-miniversal unfolding of f_0, then the unfolding of g_0 defined in lemma 2.10 is $\mathcal{K}_V$-versal so the inequality is reversed. We conclude,

Theorem 1: *With the preceding notation*

$$\mathcal{A}_e\text{-codim}(f_0) = \mathcal{K}_{V,e}\text{-codim}(g_0) .$$

§3. Isomorphism of Normal Spaces

As in the preceding section, we let $f_0 : \mathbb{C}^n,0 \longrightarrow \mathbb{C}^p,0$ have a stable unfolding $F : \mathbb{C}^{n'},0 \longrightarrow \mathbb{C}^{p'},0$ with $g_0 : \mathbb{C}^p,0 \longrightarrow \mathbb{C}^{p'},0$ denoting the inclusion of $\mathbb{C}^p$. By a choice of local coordinates we may assume $F(x,u) = (\bar{F}(x,u),u) = (y,u)$ and $g_0(y) = (y, 0)$.

In this section we shall define an isomorphism between $N\mathcal{K}_{V,e}\cdot g_0$ and $N\mathcal{A}_e\cdot f_0$ when both (i.e. either) are finite dimensional. For $\zeta \in \theta(g_0)$, we may represent $\zeta = (\zeta_1, \zeta_2)$ where ζ_1 denotes the y-component and ζ_2 the u-component of ζ. We define a $\mathcal{C}_y$-linear homomorphism $\Phi : \theta(g_0) \to \theta(f_0)$ by

$$\Phi(\zeta) = -\zeta_1 \circ f_0 + d_u\bar{F}(x,0)(\zeta_2) \circ f_0$$

Theorem 2: Φ *induces an isomorphism of* $\mathcal{C}_y$*-modules*

$$\bar{\Phi} : N\mathcal{K}_{V,e}\cdot g_0 \xrightarrow{\sim} N\mathcal{A}_e\cdot f_0$$

Proof: The proof of this theorem will occupy the rest of this section.

Given a 1-parameter unfolding of g_0, which we denote by $g_t(y)$ instead of $\bar{g}(y,t)$, we can associate to it an element of $\theta(g_0)$, namely $\zeta = \frac{\partial g_t}{\partial t}|_{t=0}$. We shall explicitly show that $\Phi(\zeta)$ is the corresponding element of $\theta(f_0)$ obtained from the induced deformation f_t of f_0 which is defined as a fiber product

(3.1) $$f_t : X_t = \{(x,u,y) : F(x,u) = g_t(y)\} \xrightarrow{pr} \mathbb{C}^p$$

with $pr(x,u,y) = y$.

We write $g_t(y) = (g_{1t}(y), g_{2t}(y)) = (y,u)$ so that $g_{10}(y) = y$, $g_{20}(y) \equiv 0$. Then, (3.1) defines X_t by

$$\bar{F}(x,u) = g_{1t}(y) \quad \text{and} \quad u = g_{2t}(y);$$

or x and y are related by

(3.2) $$g_{1t}(y) - \bar{F}(x, g_{2t}(y)) = 0.$$

Let $H(x,y,t)$ denote the function on the left hand side of (3.2). We apply the implicit function theorem to parametrize $H^{-1}(0)$.

(3.3) $$d_yH(0,0,0) = d_y g_{10}(0) - d_u\bar{F}(0,0)\circ dg_{20}(0).$$

Since $g_{10} = \text{id}$ and $g_{20} \equiv 0$, we see from (3.3) that $d_yH(0,0,0) = I$. Thus, by the implicit function theorem, we may represent $H^{-1}(0)$ as the graph of y as a function of (x,t), $y = \psi_t(x)$.

Then, $X_t = \{(x,u,y) : u = g_{2t}(y)\ ,\ y = \psi_t(x)\}$. Let $\varphi_t(x) = g_{2t} \circ \psi_t(x)$ so that $\varphi_0(x) = g_{20} \circ \psi_0(x) \equiv 0$. Also, $g_{10} = \mathrm{id}$ so for small t , g_{1t} is a germ of a diffeomorphism. Hence, by (3.2)

$$y = g_{1t}^{-1} \circ \bar{F}(x, g_{2t}(y)) .$$

Thus, by the above description of X_t and (3.1),

$$y = \psi_t(x) = g_{1t}^{-1} \circ \bar{F}(x, \varphi_t(x))$$

and so

(3.4) $$f_t(x) = g_{1t}^{-1} \circ \bar{F}(x, \varphi_t(x)) .$$

Thus, by the chain rule

(3.5) $$\frac{\partial f_t}{\partial t}\Big|_{t=0} = \frac{\partial g_{1t}^{-1}}{\partial t}\Big|_{t=0} \circ \bar{F}(x,\varphi_0(x)) + dg_{10}^{-1} \circ \frac{\partial \bar{F}}{\partial u}(x,\varphi_0(x)) \frac{\partial \varphi_t}{\partial t}\Big|_{t=0} .$$

From $g_{1t}^{-1} \circ g_{1t} = \mathrm{id}$ we obtain

(3.6) $$\frac{\partial g_{1t}^{-1}}{\partial t}\Big|_{t=0} \circ g_{10} + dg_{10}^{-1} \frac{\partial g_{1t}}{\partial t}\Big|_{t=0} = 0 .$$

Since $g_{10} = \mathrm{id}$, (3.6) implies

$$\frac{\partial g_{1t}^{-1}}{\partial t}\Big|_{t=0} = -\frac{\partial g_{1t}}{\partial t}\Big|_{t=0}$$

Also, $\varphi_0(x) = 0$ and $\bar{F}(x,0) = f_0(x)$ so (3.5) becomes

(3.7) $$\frac{\partial f_t}{\partial t}\Big|_{t=0} = -\frac{\partial g_{1t}}{\partial t}\Big|_{t=0} \circ f_0(x) + \frac{\partial \bar{F}}{\partial u}(x,0) \frac{\partial \varphi_t}{\partial t}\Big|_{t=0} .$$

Then,

$$\varphi_t = g_{2t} \circ \psi_t ,\ \text{or}$$

$$\varphi_t(x) = g_{2t} \circ g_{1t}^{-1} \circ \bar{F}(x, \varphi_t(x)) .$$

Hence

(3.8) $$\frac{\partial \varphi_t}{\partial t}\Big|_{t=0} = \frac{\partial g_{2t}}{\partial t}\Big|_{t=0} \circ g_{10}^{-1} \circ \bar{F}(x,\varphi_0(x)) + dg_{20} \circ (-) .$$

Since g_{20} , and hence dg_{20} , equals 0 , the second term vanishes. Thus, (3.8) becomes

$$\frac{\partial \varphi_t}{\partial t}\Big|_{t=0} = \frac{\partial g_{2t}}{\partial t}\Big|_{t=0} \circ f_0(x) .$$

Substituting into (3.7) yields

$$\text{(3.9)}\qquad \frac{\partial f_t}{\partial t}\Big|_{t=0} = -\frac{\partial g_{1t}}{\partial t}\Big|_{t=0} \circ f_0(x) + d_u\bar{F}(x,0)\left(\frac{\partial g_{2t}}{\partial t}\Big|_{t=0}\right)\circ f_0(x)\ .$$

If

$$\zeta = \frac{\partial g_t}{\partial t}\Big|_{t=0} = \left(\frac{\partial g_{1t}}{\partial t}\Big|_{t=0}, \frac{\partial g_{2t}}{\partial t}\Big|_{t=0}\right) = (\zeta_1, \zeta_2)$$

then, we obtain from (3.9)

$$\text{(3.10)}\qquad \frac{\partial f_t}{\partial t}\Big|_{t=0} = -\zeta_1 \circ f_0 + d_u\bar{F}(x,0)\,(\zeta_2)\circ f_0\,.$$

We see that $\Phi(\zeta)$ is equal to the right hand side of (3.10):

$$\text{(3.11)}\qquad \begin{gathered} \Phi : \theta(g_0) \longrightarrow \theta(f_0) \\ \Phi(\zeta) = -\zeta_1 \circ f_0 + d_u\bar{F}(x,0)\,(\zeta_2)\circ f_0 \end{gathered}$$

Next, if $\zeta \in T\mathcal{K}_{V,e}\cdot g_0$, then $\zeta = \frac{\partial g_t}{\partial t}\Big|_{t=0}$ for g_t a $\mathcal{K}_V$-trivial deformation. By proposition 2.4, f_t is an $\mathcal{A}$-trivial deformation of f_0. Thus,

$$\Phi(\zeta) = \frac{\partial f_t}{\partial t}\Big|_{t=0} \in T\mathcal{A}_e\cdot f_0\,.$$

Thus,

$$\Phi(T\mathcal{K}_{V,e}\cdot g_0) \subset T\mathcal{A}_e\cdot f_0$$

and induces a $\mathcal{C}_y$-module homomorphism.

$$\text{(3.12)}\qquad \bar{\Phi} : N\mathcal{K}_{V,e}\cdot g_0 \longrightarrow N\mathcal{A}_e\cdot f_0\,.$$

We now show this is an isomorphism.

Given $\xi \in \theta(f_0)$, then $\xi = \frac{\partial f_t}{\partial t}\Big|_{t=0}$ with f_t induced, up to $\mathcal{A}$-equivalence, by an unfolding g_t. Thus, g_t induces f'_t with f_t $\mathcal{A}$-equivalent to f'_t, say $f_t = \psi_t \circ f'_t \circ \varphi_t$ with $\psi_0 =$ id, $\varphi_0 =$ id. We compute

$$\frac{\partial f_t}{\partial t}\Big|_{t=0} = \frac{\partial f'_t}{\partial t}\Big|_{t=0} + \frac{\partial \psi_t}{\partial t}\Big|_{t=0} \circ f'_0 - df'_0\left(\frac{\partial \varphi_t}{\partial t}\Big|_{t=0}\right)$$

$$= \frac{\partial f'_t}{\partial t}\Big|_{t=0} + \eta \circ f_0 - \xi(f_0) \ .$$

Thus,

$$\xi = \frac{\partial f_t}{\partial t}\Big|_{t=0} \equiv \Phi(\zeta) \bmod T\mathcal{A}_e \cdot f_0 \, .$$

Hence, $\bar{\Phi}$ is surjective.

By Theorem 1, the spaces in (3.12) have the same dimension as vector spaces; as $\bar{\Phi}$ is surjective it is an isomorphism. □

We can now refine our earlier results relating the versality of g_0 and f_0.

Corollary 1: *With the preceding notation, let* g *be an unfolding of* g_0 *and let* f *denote the induced unfolding of* f_0. *Then, f is* $\mathcal{A}$-*versal if and only if* g *is* $\mathcal{K}_V$-*versal.*

Proof: The proof of the theorem shows that for each i,

$$\bar{\Phi}\left(\frac{\partial g}{\partial u_i}\right) = \frac{\partial f}{\partial u_i}$$

Hence,the corollary follows by the versality theorem and theorem 2. □

We also obtain the analog of theorem 2 for multi-germs, which follows by the same proofs except applied to multi-germs.

Let $f_0: \mathbb{C}^n,S \longrightarrow \mathbb{C}^p,0$ have a stable unfolding $F : \mathbb{C}^{n'},S \longrightarrow \mathbb{C}^{p'},0$ with $g_0 : \mathbb{C}^p,0 \longrightarrow \mathbb{C}^{p'},0$ denoting the inclusion of $\mathbb{C}^p$. Then, Φ defined by (3.11) also defines a homomorphism for $\theta(f_0)$ denoting the module of vector fields along the multi-germ f_0. Then, Φ also induces an isomorphism in this case.

Theorem 3: i) *The multi-germ* f_0 *has finite* $\mathcal{A}$-*codimension if and only if* g_0 *has finite* $\mathcal{K}_V$-*codimension;*

ii) *in the case of* i) Φ *induces an isomorphism*

$$\bar{\Phi} : N\mathcal{K}_{V,e} \cdot g_0 \xrightarrow{\sim} N\mathcal{A}_e \cdot f_0 .$$

§4 Several Consequences

We deduce consequences of the main theorems for: a) placing Mond's formula in a more general context as an analogue of Milnor's formula but for nonlinear sections of nonisolated hypersurface singularities and b) verifying that a method for computing the versality discriminant of an unfolding of a hypersurface singularity (given in [DG] for the Pham example) is valid in general.

Nonlinear Sections of Hypersurface Singularities

Let $V,0 \subset \mathbb{C}^m,0$ be a hypersurface germ and let $g_0 : \mathbb{C}^p,0 \longrightarrow \mathbb{C}^m,0$ be a germ of an immersion. We can define two numbers associated to the nonlinear section g_0, a number defined algebraically, which measures the codimension of g_0, and a number defined geometrically, which is the analogue of the Milnor number for $g_t(\mathbb{C}^p) \cap V$ with g_t a perturbation of g_0. If we ask when these two numbers are equal, it turns out that not only can Mond's formula be interpreted as an equality of these numbers but, in this context, it is related to other formulas which involve seemingly unrelated numbers such as the multiplicity of the discriminant for a versal deformation and a special case of Greuel's and Lê's formula for the Milnor number of isolated complete intersection singularities [G], [L].

The algebraically defined number associated to g_0 is its $\mathcal{K}_V$-codimension

$$\nu_{alg}(g_0) = \mathcal{K}_{V,e}\text{-codim}(g_0)$$

For this number to be finite we must assume that g_0 is transverse to V in a punctured neighborhood of 0. For the geometrically defined number, we consider a one-parameter family of germs g_t such that g_t is transverse to V for $t \neq 0$. Here we have to use a weaker notion of transversality than that used in § 1, i.e. we choose a Whitney stratification of V with the property that all $\eta \in \theta_V$ are tangent to the strata and require transversality to all of the strata. Then the geometric number which is the analogue of the Milnor number is

$$\nu_{geom}(g_0) = |\chi(g_t(\mathbb{C}^p) \cap V \cap B_\varepsilon) - 1|.$$

Here $\chi(g_t(\mathbb{C}^p) \cap V \cap B_\varepsilon)$ is the topological Euler characteristic, B_ε is a ball about 0 of radius ε and ε and t have to chosen appropriately small. This geometric number can be shown to be well-defined. Mond's formula and other related formulas suggest the following.

BASIC QUESTION : Suppose both V and g_0 are weighted homogeneous for the same weights on $\mathbb{C}^m$. When do we have the analogue of Milnor's formula, namely, when does (4.1) hold?

(4.1) $$\nu_{alg}(g_0) = \nu_{geom}(g_0)$$

We consider some cases where it is presently known to hold.

1) Let $V = D(F) = \text{image}(F)$ where $F: \mathbb{C}^n,0 \longrightarrow \mathbb{C}^{n+1},0$ is a stable germ, and let $g_0: \mathbb{C}^3,0 \longrightarrow \mathbb{C}^{n+1},0$ denote a germ of an immersion transverse to F with f_0 the pullback. By theorem 2

$$\mathcal{A}_e\text{-codim}(f_0) = \mathcal{K}_{V,e}\text{-codim}(g_0) = \nu_{alg}(g_0).$$

If g_t is a family such that $g_t(\mathbb{C}^3)$ is transverse to V for $t \neq 0$, then by the proof of proposition 2.3, the pull-back family f_t is stable for $t \neq 0$. Then $f_t(\mathbb{C}^2) \cap B_\varepsilon = g_t(\mathbb{C}^3) \cap V \cap B_\varepsilon$. Thus, $\nu_{geom}(g_0) = |\chi(f_t(\mathbb{C}^2) \cap B_\varepsilon) - 1|$. Thus, by the result of de Jong and van Straten [JS], (4.1) holds when g_0 and F are weighted homogeneous for the same weights on $\mathbb{C}^{n+1}$.

2) Let $V = D(F)$ where $F: \mathbb{C}^{n+q},0 \longrightarrow \mathbb{C}^{1+q},0$ is a versal unfolding of a weighted homogeneous hypersurface singularity defined by f_0 (here $q = \tau - 1$). Also, let $g_0: \mathbb{C},0 \longrightarrow \mathbb{C}^{1+q},0$ denote the germ $g_0(y) = (y,0)$. Then, g_0 is transverse to F with f_0 the pullback. We saw in example 1.3 that

$$\mathcal{A}_e\text{-codim}(f_0) = \mathcal{K}_{V,e}\text{-codim}(g_0) \ (= \nu_{alg}(g_0)) = \tau - 1$$

If g_t is transverse to V for t small $\neq 0$ then $g_t(\mathbb{C}) \cap V \cap B_\varepsilon$ consists of m(V) points, where m(V) denotes the multiplicity of V. However, $m(V) = \mu$; see e.g. [T]. Hence, $\nu_{geom}(g_0) = \mu - 1$. Since f_0 is weighted homogeneous, $\mu = \tau$ so again we have equality in (1.4).

3) Let $V = D(F) = \text{image}(F)$ where $F: \mathbb{C}^n,0 \longrightarrow \mathbb{C}^{n+1},0$ is the stable unfolding of the germ $f_0(x) = (x^n,x^m)$ with $(n,m) = 1$. Likewise, let $g_0: \mathbb{C}^2,0 \longrightarrow \mathbb{C}^{n+1},0$ denote the germ of an immersion $g_0(y_1,y_2) = (y_1,y_2,0)$ transverse to F with f_0 the pullback. Then, a simple calculation shows that $\mathcal{A}_e\text{-codim}(f_0) = \delta(C)$ where C is the image curve $f_0(\mathbb{C})$ defined by $y_1{}^m - y_2{}^n = 0$. Also, f_0 can be deformed to a stable germ f_t so that the image curve $f_t(\mathbb{C})$ has $\delta(C)$ double points and $f_t(\mathbb{C}) \cap B_\varepsilon = g_t(\mathbb{C}^2) \cap V \cap B_\varepsilon$ for g_t the deformation of g_0 inducing f_t. Hence,

$$\nu_{geom}(g_0) = |\chi(f_t(\mathbb{C}) \cap B_\varepsilon) - 1| = |(1 - \delta) - 1| = \delta(C).$$

Lastly we consider a hypersurface which is not the discriminant of a stable germ.

4) Let V,0 be an isolated hypersurface singularity defined by a weighted homogeneous germ $f_0 : \mathbb{C}^n,0 \longrightarrow \mathbb{C},0$. Let $g_0: \mathbb{C}^{n-1},0 \longrightarrow \mathbb{C}^n,0$ be a germ of an immersion which is weighted homogeneous for the same weights and for which $g_0(\mathbb{C}^{n-1})$ is transverse to V in a punctured neighbourhood of 0. By a weighted homogeneous change of coordinates we may assume that g_0 is a linear embedding $g_0(x_1, \dots ,x_{n-1}) = (x_1, \dots ,x_{n-1},0)$. Now, θ_V is generated by $\{\zeta_{ij} = \frac{\partial f_0}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial f_0}{\partial x_j}\frac{\partial}{\partial x_i}, e\}$ where e is the Euler vector field. Thus,

$$N\mathcal{K}_{V,e}\cdot g_0 = \mathcal{C}_{X'}\left\{\frac{\partial}{\partial x_i}\right\} / (\mathcal{C}_{X'}\left\{\frac{\partial}{\partial x_j}, j= 1,\dots , n-1\right\} + \mathcal{C}_{X'}\left\{\zeta_{ij}, e\right\}$$

$$= \mathcal{C}_{X'}\left\{\frac{\partial}{\partial x_n}\right\} / (\mathcal{C}_{X'}\left\{\frac{\partial f_0}{\partial x_j}\cdot\frac{\partial}{\partial x_n}, j= 1,\dots , n-1\right\}$$

$$\text{(since } e \circ g_0 = \sum_{j=1}^{n-1} x_j\cdot\frac{\partial}{\partial x_j}\text{)}$$

$$= \mathcal{C}_{X'} / (\frac{\partial f_0}{\partial x_j}|_{x_n=0}, j= 1,\dots , n-1).$$

Therefore,

$$\mathcal{K}_{V,e}\text{-codim}(g_0) = \mu(f_0|\mathbb{C}^{n-1}).$$

On the other hand, the assumption of transversality of $\mathbb{C}^{n-1}$ to V off of 0 implies that $h = (x_n,f_0) : \mathbb{C}^n,0 \longrightarrow \mathbb{C}^2,0$ defines an isolated weighted homogeneous complete intersection singularity. For $g_t(x') = (x', t)$, $g_t(\mathbb{C}^{n-1})$ is transverse to V for small $t \neq 0$. Thus, $h^{-1}(0,t) = g_t(\mathbb{C}^{n-1}) \cap V$ is a Milnor fiber of h so that $\nu_{geom}(g_0) = \mu(h) = \mu(f_0|\mathbb{C}^{n-1})$ (e.g. by a result of Greuel and Lê, [G] [L], $= \dim_{\mathbb{C}}\mathcal{J}(h)$, where $\mathcal{J}(h)$ is the Jacobian algebra of h, and by direct computation,

$$\mathcal{J}(h) = \mathcal{C}_X/(\frac{\partial f_0}{\partial x_j}|_{x_n=0}, j= 1,\dots , n-1)).$$

Again (1.4) holds.

Versality Discriminant

Let $f_0 : \mathbb{C}^n,0 \longrightarrow \mathbb{C},0$ be a weighted homogeneous isolated hypersurface singularity. We can assign weights to $\theta(f_0)$ via $wt(\frac{\partial}{\partial x_i}) = -wt(x_i)$ and this induces weights on $N\mathcal{A}_e \cdot f_0 = N\mathcal{K}^+{}_e \cdot f_0$ ($\mathcal{K}^+$ is the usual action of $\mathcal{K}$ together with $\mathbb{C}$ acting by translation on $\mathbb{C}$). We let $N\mathcal{A}_e \cdot f_{0(<m)}$ denote the terms of weight < m. For a given m, let $\{\varphi_i\}_{i=1}^q$ be a basis for $N\mathcal{A}_e \cdot f_{0(<m)}$ and consider the unfolding

$$F(x,u) = (\bar{F}(x,u), u) = (f_0(x) + \sum_{i=1}^{q} u_i\varphi_i, u)$$

The versality discriminant for F consists of $z = (y,u) \in \mathbb{C}^{1+q}$ such that if $S = F^{-1}(y,u) \cap \Sigma(F)$ (recall $\Sigma(F)$ is the critical set of F) then $F : \mathbb{C}^{n+q},S \longrightarrow \mathbb{C}^{1+q},z$ is not infinitesimally stable. Understanding the versality discriminant of F is a basic step in understanding the structure of F and determining whether, e.g., it is topologically versal.

In [DG] a procedure was given for computing the versality discriminant for the Pham example. We show here that this procedure works for all such unfoldings described above. Let $\{\bar{\varphi}_i\}_{i=1}^r$ be a basis for $N\mathcal{A}_e \cdot f_{0(\geq m)}$. Also, let

$$e = d.y.\frac{\partial}{\partial y} - \sum_{j=1}^{n} a_j x_j \frac{\partial}{\partial x_j} \qquad \text{with } a_j = wt(x_j) \text{ and } d = wt(y).$$

Then,

$$e(\bar{F}) \overset{\text{def}}{=} d.\bar{F}.\frac{\partial}{\partial y} - \sum_{j=1}^{n} a_j x_j . \frac{\partial \bar{F}}{\partial x_j} = \sum_{j=1}^{q} wt(u_j) \cdot u_j \cdot \varphi_j$$

Since $\{\varphi_i , \bar{\varphi}_j\}$ is a basis for $N\mathcal{A}_e . f_0$, by the preparation theorem we may write

$$\varphi_i \cdot e(\bar{F}) = \sum_{j=1}^{q} h_{ij}(u) \cdot \varphi_j + \sum_{j=1}^{r} g_{ij}(u) \cdot \bar{\varphi}_j \mod \left(\frac{\partial \bar{F}}{\partial x_1}, \dots, \frac{\partial \bar{F}}{\partial x_n}\right)$$

(4.2)

$$\bar{\varphi}_i \cdot e(\bar{F}) = \sum_{j=1}^{q} h'_{ij}(u) \cdot \varphi_j + \sum_{j=1}^{r} g'_{ij}(u) \cdot \bar{\varphi}_j \mod \left(\frac{\partial \bar{F}}{\partial x_1}, \dots, \frac{\partial \bar{F}}{\partial x_n}\right)$$

Let H be the $(\tau-1)\times r$ matrix with entries

$$H = \big(\underset{q\times r}{g_{ij}(u)} \mid \underset{r\times r}{y\cdot\delta_{ij} - g'_{ij}(u)}\big)$$

Let W be the variety defined by the vanishing of the $r \times r$ minors of H. For the Pham example [DG], it is shown that this yields the versality discriminant. This is in fact true in general:

Proposition 4.3: W *is the versality discriminant of* F.

Proof: The proof is a consequence of the proof of proposition 2.3 together with the construction due to Saito [S] of the generators of θ_V for $V = D(F_1)$ with F_1 the versal unfolding of f_0 (see example 1.3)

$$F_1(x,u.v) = (f_0(x) + \sum_{i=1}^{q} u_i\cdot\varphi_i + \sum_{j=1}^{r} v_j\cdot\bar{\varphi}_j, u, v)$$

By the proof of of proposition 2.3, $(y,u) \in \mathbb{C}^{1+q} \subset \mathbb{C}^{1+q+r}$ belongs to W exactly if $\mathbb{C}^{1+q}$ fails to be transverse to $V = D(F_1)$ at $(y,u,0)$. Recall that transversality holds at (y,u) if

$$\mathbb{C}^{1+q} + \langle \eta_{0(y,u)}, \ldots, \eta_{\tau-1(y,u)} \rangle = \mathbb{C}^{1+q+r} \tag{4.4}$$

where $\{\eta_i\}$ denote the set of generators for θ_V constructed by Saito. However, note that (4.2) implies

$$\begin{aligned} d\cdot\bar{F}\cdot\varphi_i\cdot\frac{\partial}{\partial y} &= \sum_{j=1}^{q} h_{ij}(u)\cdot\varphi_j + \sum_{j=1}^{r} g_{ij}(u)\cdot\bar{\varphi}_j \mod\left(\frac{\partial\bar{F}}{\partial x_1},\ldots,\frac{\partial\bar{F}}{\partial x_n}\right) \\ d\cdot\bar{F}\cdot\bar{\varphi}_i\cdot\frac{\partial}{\partial y} &= \sum_{j=1}^{q} h'_{ij}(u)\cdot\varphi_j + \sum_{j=1}^{r} g'_{ij}(u)\cdot\bar{\varphi}_j \mod\left(\frac{\partial\bar{F}}{\partial x_1},\ldots,\frac{\partial\bar{F}}{\partial x_n}\right) \end{aligned} \tag{4.5}$$

It follows from Saito's construction (1.3) and (4.5) that

$$\eta_k = (y-h_{kk})\cdot\frac{\partial}{\partial u_k} + \sum_{j\neq k} h_{kj}(u)\cdot\frac{\partial}{\partial u_j} + \sum_{j=1}^{r} g_{kj}(u)\cdot\frac{\partial}{\partial v_j} \qquad 1 \le k \le q$$

$$\eta_{q+k} = (y-g'_{kk})\cdot\frac{\partial}{\partial v_k} + \sum_{j=1}^{q} h_{kj}(u)\cdot\frac{\partial}{\partial u_j} + \sum_{j\neq k} g_{kj}(u)\cdot\frac{\partial}{\partial v_j} \qquad 1 \le k \le r$$

together with

$$\eta_0 = d.y.\frac{\partial}{\partial y} - \sum_{j=1}^{q} b_j\cdot u_j\cdot\frac{\partial}{\partial u_i} + \sum_{j=1}^{r} c_j\cdot v_j\cdot\frac{\partial}{\partial v_j}$$

are generators for V. Hence, (4.4) holds exactly when the v-components of the vectors $\eta_j | \mathbb{C}^{1+q}$ span $\mathbb{C}^r$. These components give (up to signs in columns) exactly the matrix H (since $\eta_0 | \mathbb{C}^{1+q}$ has v-component = 0). Thus, (4.4) fails exactly when the rank of H < r. □

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